Metric Spaces and Topology
Lecture 1

Prerequisites. The set of reals $\mathbb{R}$, inf / sup and their existence for bounded sets (order-completeness).

Def. Let X be a set. A *metric* on X is a function $d : X \times X \to [0,\infty)$ such that

for all ... implies

(i) $d(x,x) = 0 \quad \forall x \in X$ (i') $\left(d(x,y) = 0 \Longrightarrow x = y\right) \ \forall x,y \in X$.

(ii) Symmetry: $d(x,y) = d(y,x) \quad \forall x,y \in X$.

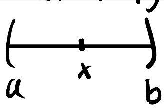
(iii) Triangle inequality (Δ-inequality): $d(x,z) \leq d(x,y) + d(y,z)$ $\forall x,y,z \in X$.

The set X equipped with a metric d is called a *metric space* (X,d).
For $r \geq 0$ and $x \in X$, $B_r^d(x) := \{ y \in X : d(x,y) < r \}$ is called the (open) *ball* at x of radius r. Omit the superscript d if it's clear.

Examples and non-examples. ○ $\mathbb{R}$ with the usual metric $d(x,y) := |x-y|$.

Check that this is a metric.

The open balls in this metric space are precisely the bounded open intervals. Indeed, $(a,b) = \left(x - \frac{b-a}{2}, x + \frac{b-a}{2}\right)$, where $x := \frac{a+b}{2}$.



$= B_{\frac{b-a}{2}}(x)$.

○ $\mathbb{R}$ or any other set X with the discrete metric:

$$d(x,y) := \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise (o.w.)} \end{cases}$$

○ $\mathbb{R}$ with $\tilde{d}(x,y) := \sqrt{|x-y|}$. This is a metric indeed:

$$\sqrt{|x-y|} + \sqrt{|y-z|} \geq \sqrt{|x-z|} \iff \left(\sqrt{|x-y|} + \sqrt{|y-z|}\right)^2 \geq |x-z|.$$

Proof. $\left(\sqrt{|x-y|} + \sqrt{|y-z|}\right)^2 = |x-y| + |y-z| + 2\sqrt{\cdots}\sqrt{\cdots} \geq |x-y| + |y-z|$
$\geq |x-z|.$ □

○ $\mathbb{R}$ with $\tilde{\tilde{d}}(x,y) := |x-y|^2$. This is not a metric because

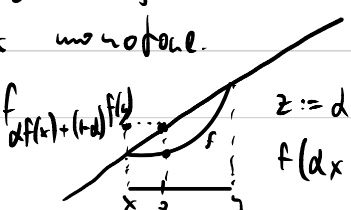
$$|x-z|^2 = |x-y+y-z|^2 \underset{\text{if } z > y > x}{=} \left(|x-y| + |y-z|\right)^2 = |x-y|^2 + |y-z|^2 + 2|x-y||y-z|$$
$$> |x-y|^2 + |y-z|^2.$$

Thus, the Δ-inequality fails.

HW*: Form a conjecture as to for which functions $f: [0,\infty) \to [0,\infty)$ it is true that if d is a metric on a set X, then so is $f(d)$. Prove your conjecture.

Hint. Concave / convex monotone.

Convexity. $f: \mathbb{R} \to \mathbb{R}$ is convex (concave) if

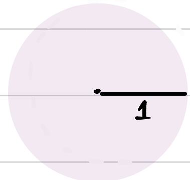


$z := \alpha x + (1-\alpha) y$, $\alpha \in [0,1]$.

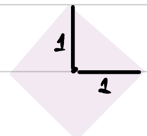
$$f(\alpha x + (1-\alpha) y) \underset{(\geq)}{\leq} \alpha f(x) + (1-\alpha) f(y).$$

- $\mathbb{R}^2$ with $d_2(x,y) := \sqrt{|x_1-y_1|^2+|x_2-y_2|^2}$, $x=(x_1,x_2)$, $y=(y_1,y_2)$, the Euclidean distance.

This is a unit ball in $\mathbb{R}^2$ with d_2:

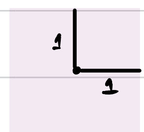


$d_1(x,y) := |x_1-y_1|+|x_2-y_2|$. $\quad B_1^{d_1}(x) :=$

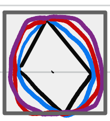


This called the New York distance

$d_\infty(x,y) := \max\left(|x_1-y_1|, |x_2-y_2|\right)$, $\quad B_1^{d_\infty}(x) :=$



$\forall\, p \geq 1$, $d_p(x,y) := \left(|x_1-y_1|^p+|x_2-y_2|^p\right)^{\frac{1}{p}}$ HW*: Show that this is a metric.



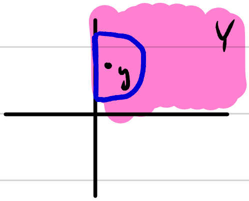
$p=1$ $\quad p=\infty$.
$p=\frac{3}{2}$
$p=2$
$p=3$

HW*: Show that $\lim_{p\to\infty} d_p = d_\infty$.

- If (X,d) is a metric space and $Y \subseteq X$, then the restriction of d to $Y\times Y$ is a metric on Y, making $(Y, d|_Y)$ is a metric space. We abuse the notation and just write (Y,d).

For $y \in Y$, the ball $B_r^{d|_Y}(y)$ is just $B_r^d(y) \cap Y$.

For example: $X := \mathbb{R}^2$, $Y := [0,\infty) \times [0,\infty)$.



$B_1^{d|_Y}(y)$.